Electromagnetic Potentials and Field Expansions for Plasma Radiation in Waveguides

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Abstract—**In order to calculate the raditaion from plasmas placed in waveguides it is necessary to know the field produced by arbitrarily moving charged particles in a waveguide. In this paper modal expansions for the vector and scalar potentials due to arbitrarily moving charged particles in a waveguide are derived and provide the extension of the Liénard-Wiechert potentials to a waveguide environment. In addition, for a plasma filled waveguide, a modal expansion is given of the electric field directly in terms of mode coupling with the charge motion. Expressions for the spectral distribution of the radiation are given, both in general and for cyclotron radiation. Some specific results for the H_{10} mode excited in a rectangular guide by cyclotron motion are also presented.**

I. Introduction

THE PROPERTIES of the electromagnetic radiation from a plasma are of great interest because they provide information about the physical process going on in a plasma and also because they are an important loss mechanism in thermonuclear machines, [1], [3]. The two most important radiation mechanisms are electron bremsstrahlung and cyclotron radiation. The main features of this type of radiation may be calculated by means of classical radiation theory. A common procedure is to first calculate the radiation from a single particle. The radiation from the whole plasma is then obtained by averaging the radiated power over the velocity distribution of the electrons, using the assumption that each electron radiates incoherently with respect to all of the others [4].

The radiation from a single accelerated charge in free space is readily found from the well-known Liénard-Wiechert potentials [5]. The radiation from a single gyrating electron in a uniform static magnetic field may be found by an application of these potentials [6], [7]. Although a number of analyses and calculations for single particle radiation in free space have been carried out, general results for arbitrarily moving charged particles in a waveguide do not seem to have been derived. The radiation from charged particles in a wave guide is of considerable importance in connection with the measurement of plasma radiation at microwave frequencies. In the measurement of the plasma radiation it is often convenient to place the plasma inside of a waveguide [8], [9]. It then becomes important to know how various types of charge motion couple to or excite propagating modes in the waveguide.

The first part of this paper is concerned with the modal expansion of the vector and scalar potentials arising from a single arbitrarily moving charge in a waveguide. The solution presented may be regarded as an extension of the Liénard-Wiechert potentials to a waveguide environment. Particular attention is focused on the cyclotron radiation. Suitable formulas are obtained for calculating the spectral density distribution of the radiation when the particle undergoes periodic motion for a finite time interval only.

The second part of the paper deals with the problem of the radiation from a single particle in a waveguide partially filled with a plasma medium. A formal solution is given for the modal expansion of the electric field in the partially filled guide In the present case the complexity of the equations satisfied by the potentials is such that it is often more convenient to work directly with the modal expansions of the electric and magnetic fields.

II. Potential Expansions for a Single Arbitrarily Moving Charge

In this section the modal expansions for the vector and scalar potentials arising from a single arbitrarily moving charge will be derived. The charged particle's trajectory is assumed to be unaffected by its own radiation field, an assumption that holds to a high degree of approximation except for extremely high energy particles (in case of electrons radiation damping is negligible for energies below one beV). It is further assumed that the particle moves in an otherwise empty waveguide. The waveguide has an arbitrary cross section, is infinitely long, and is perfectly conducting

Consider a particle with charge q and position vector $r'(t)$. The charge density ρ and current density J associated with this moving charge may be expressed as

$$\rho(r, t) = q\delta[r - r'(t)] \tag{1a}$$

$$J(r, t) = q\frac{dr'}{dt}\delta[r - r'(t)] \tag{1b}$$

where δ is the Dirac delta function. The vector potential A and scalar potential Φ are solutions of

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$$\nabla^2 A(r,t) - \frac{1}{c^2}\frac{\partial^2 A(r,t)}{\partial t^2} = -\mu_0 J(r,t) \quad (2)$$

$$\nabla^2 \Phi(r,t) - \frac{1}{c^2}\frac{\partial^2 \Phi(r,t)}{\partial t^2} = -\frac{\rho(r,t)}{\epsilon_0}. \quad (3)$$

Vector Potential Expansion

The aforementioned equations will be reduced to standard eigenfunction-eigenvalue problems by taking Fourier transforms with respect to time t and the waveguide axial coordinate z. Thus we let

$$A(r,\omega) = \int_{-\infty}^{\infty} A(r,t)e^{-j\omega t}dt \quad (4a)$$

$$A(r_t,\beta,\omega) = \int_{-\infty}^{\infty} A(r,\omega)e^{j\beta z}dz \quad (4b)$$

where r_t is the x and y part of r. Similar definitions are used for the transforms of all other quantities. The Fourier transform of (2) yields

$$\begin{aligned}(\nabla_t^2 + k_c^2)&A(r_t,\beta,\omega)\\ &= -\mu_0 J(r_t,\beta,\omega)\\ &= -\mu_0 q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dr'(t')}{dt'}\delta[r_t - r_t'(t')]\cdots\\ &\quad\cdot\delta[z - z'(t')]e^{-j\omega t'}e^{j\beta z}dt'dz\\ &= -\mu_0 q\int_{-\infty}^{\infty}\frac{dr'(t')}{dt'}\delta[r_t - r_t'(t')]e^{-j\omega t'+j\beta z'}dt' \quad (5)\end{aligned}$$

where $k_c^2 = k_0^2 - \beta^2$ and $k_0^2 = \omega^2/c^2$. The solution to (5) will be expanded in terms of the eigenfunctions of the equations

$$\nabla_t^2 A_t(r_t) + k_c^2 A_t(r_t) = 0 \quad (6a)$$

$$\nabla_t^2 A_z(r_t) + k_c^2 A_z(r_t) = 0. \quad (6b)$$

In a gauge where the divergence is nonzero the general solution of (6a) consists of two classes of vector eigenfunctions, namely

1) Solenoidal modes with eigenvalues l_n^2 which will be designated by the symbol F_n. These satisfy the relation $\nabla_t \cdot F_n = 0$.
2) Irrotational modes with eigenvalues k_n^2 which will be denoted by the symbol G_n. These satisfy the condition $\nabla_t \times G_n = 0$.

For unique solutions, and keeping in mind that the electromagnetic fields to be derived from A must satisfy the boundary conditions

$$n \times E = n \cdot H = 0 \qquad \text{on } C$$

where C is the waveguide boundary, the following boundary conditions are imposed:

$$\nabla_t \cdot G_n = n \times F_n = 0 \qquad \text{on } C. \quad (7)$$

Solutions for the G_n are readily found in terms of scalar function $\Psi_n(r_t)$ as follows:

$$G_n = \frac{\nabla_t \Psi_n}{k_n} \quad (8)$$

where

$$\nabla_t^2 \Psi_n + k_n^2 \Psi_n = 0 \quad (9a)$$

$$\Psi_n = 0 \qquad \text{on } C. \quad (9b)$$

These functions are readily shown to be orthogonal with respect to integration over the guide cross section S and are assumed to be normalized so that for nondegenerate eigenvalues

$$\int_S G_n \cdot G_m dS = \int_S \Psi_n \Psi_m dS = \delta_{nm} \quad (10)$$

where $\delta_{nm} = 0$ for $n \neq m$ and equals unity for $n = m$. Degenerate eigenfunctions may be combined in a linear fashion so that (10) will be valid for these as well. It is clear that the functions G_n as given satisfy the conditions $\nabla_t \times G_n = 0$ and $\nabla_t \cdot G_n = 0$ on C.

Solution for the F_n may be obtained in terms of scalar functions $\phi_n(r_t)$ as follows:

$$F_n = \frac{1}{l_n}\nabla_t \times a_z\phi_n = -\frac{a_z \times \nabla_t \phi_n}{l_n} \quad (11)$$

where

$$\nabla_t^2 \phi_n + l_n^2 \phi_n = 0 \quad (12a)$$

$$\frac{\partial \phi_n}{\partial n} = 0 \qquad \text{on } C. \quad (12b)$$

These functions are orthogonal and are also assumed normalized so that

$$\int_S F_n \cdot F_m dS = \int_S \phi_n \phi_m dS = \delta_{nm}. \quad (13)$$

Clearly $\nabla_t \cdot F_n = 0$ and $n \times F_n = 0$ on C. It is assumed that the guide cross section is simply connected so that zero eigenvalues, i.e., $k_n = l_n = 0$, are excluded.

The axial component A_z of the vector potential may be expanded in terms of the Ψ_n functions which are also the eigenfunctions of (6b).

In addition to the orthonormal properties (10) and (13) the eigenfunction F_n and G_m are also mutually orthogonal, i.e.,

$$\int_S F_n \cdot G_m dS = 0, \qquad \text{all } n \text{ and } m. \quad (14)$$

A suitable expansion for the vector potential $A(r_t,\beta,\omega)$ in (5) is

$$A(r_t,\beta,\omega) = \sum_n f_n F_n + \sum_n g_n G_n + a_z \sum_n a_n \Psi_n \quad (15)$$

where f_n, g_n, and a_n are expansion coefficients. When this expansion is used in (5) and use is made of the orthogonality properties of the functions $\boldsymbol{F}_n$, $\boldsymbol{G}_n$, and Ψ_n, it is readily found that

$$a_n = \frac{-\mu_0 q}{k_c^2 - k_n^2} \int_{-\infty}^{\infty} \frac{dz'(t')}{dt'} \Psi_n[\boldsymbol{r}_t'(t')] e^{-j\omega t' + j\beta z'} dt' \qquad (16a)$$

$$f_n = \frac{-\mu_0 q}{k_c^2 - l_n^2} \int_{-\infty}^{\infty} \boldsymbol{F}_n[\boldsymbol{r}_t'(t')] \cdot \frac{d\boldsymbol{r}_t'(t')}{dt'} e^{-j\omega t' + j\beta z'} dt' \qquad (16b)$$

$$g_n = \frac{-\mu_0 q}{k_c^2 - k_n^2} \int_{-\infty}^{\infty} \boldsymbol{G}_n[\boldsymbol{r}_t'(t')] \cdot \frac{d\boldsymbol{r}_t'(t')}{dt'} e^{-j\omega t' + j\beta z'} dt'. \qquad (16c)$$

Note that the a_n, f_n, and g_n are functions of β and ω.

The transform relation for a_n, f_n, and g_n in (16) may be inverted with respect to β to give

$$\begin{aligned} a_n(\omega, z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} a_n(\omega, \beta) e^{-j\beta z} d\beta \\ &= \frac{-\mu_0 q}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_n[\boldsymbol{r}_t'(t')] \frac{dz'(t')}{dt'} \\ &\quad \cdot e^{-j\omega t'} \frac{e^{-j\beta(z-z')}}{\beta_n^2 - \beta^2} d\beta dt' \\ &= \frac{-\mu_0 q j}{2\beta_n} \int_{-\infty}^{\infty} \Psi_n[\boldsymbol{r}_t'(t')] \frac{dz'}{dt'} e^{-j\beta_n |z-z'|} e^{-j\omega t'} dt' \qquad (17a) \end{aligned}$$

where β_n^2 is defined by $\beta_n^2 = k_0^2 - k_n^2$. The inversion contour encircles the pole at $\beta = \beta_n$ above the axis, and the pole at $\beta = -\beta_n$ below the axis. Similarly

$$f_n(\omega, z) = \frac{-j\mu_0 q}{2p_n} \int_{-\infty}^{\infty} \boldsymbol{F}_n(\boldsymbol{r}_t') \cdot \frac{d\boldsymbol{r}_t'}{dt'} e^{-jp_n|z-z'| - j\omega t'} dt' \qquad (17b)$$

where $p_n^2 = k_0^2 - l_n^2$,

$$g_n(\omega, z) = \frac{-j\mu_0 q}{2\beta_n} \int_{-\infty}^{\infty} \boldsymbol{G}_n(\boldsymbol{r}_t') \cdot \frac{d\boldsymbol{r}_t'}{dt'} e^{-j\beta_n|z-z'| - j\omega t'} dt'. \qquad (17c)$$

In (17) $\boldsymbol{r}_t'$ and z' are explicit functions of t'. The coefficients $a_n(\omega, z)$, $f_n(\omega, z)$, and $g_n(\omega, z)$ in (17) give the spectral distribution of the n'th mode in the expansion of the vector potential. Note in particular that when $k_0^2 = \omega^2/c^2$ is less than k_n^2 or l_n^2 the mode is cut off and does not propagate.

In order to obtain the expansion of $\boldsymbol{A}(\boldsymbol{r}, t)$ the expansion coefficients $a_n(\omega, z)$, $f_n(\omega, z)$, and $g_n(\omega, z)$ as given by (17) must be inverted with respect to ω.

This requires evaluation of an integral of the form

$$I = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\exp\left[j\omega(t-t') - j \frac{|z-z'|}{c} \sqrt{\omega^2 - c^2 k_n^2} \right]}{\sqrt{\omega^2 - c^2 k_n^2}} d\omega. \qquad (18a)$$

This is a standard integral occurring in waveguide transient problems and has the value

$$\begin{aligned} I &= J_0[k_n \sqrt{c^2(t-t')^2 - (z-z')^2}], \quad t - t' \geq |z - z'|/c \\ &= 0, \qquad \text{otherwise} \qquad (18b) \end{aligned}$$

where J_0 is the Bessel function of the first kind and order zero. Use of this result gives

$$a_n(t, z) = \frac{Z_0 q}{2} \int_{-\infty}^{\infty} \Psi_n(\boldsymbol{r}_t') \frac{dz'}{dt'} J_0 \cdot [k_n \sqrt{c^2(t-t')^2 - (z-z')^2}] dt' \qquad (19a)$$

where $\boldsymbol{r}_t'$ and z' are functions of t', $Z_0 = \sqrt{\mu_0/\epsilon_0}$

$$f_n(t, z) = \frac{Z_0 q}{2} \int_{-\infty}^{\infty} \boldsymbol{F}_n(\boldsymbol{r}_t') \cdot \frac{d\boldsymbol{r}_t'}{dt'} J_0 \cdot [l_n \sqrt{c^2(t-t')^2 - (z-z')^2}] dt'$$

$$g_n(t, z) = \frac{Z_0 q}{2} \int_{-\infty}^{\infty} \boldsymbol{G}_n(\boldsymbol{r}_t') \cdot \frac{d\boldsymbol{r}_t'}{dt'} J_0 \cdot [k_n \sqrt{c^2(t-t')^2 - (z-z')^2}] dt'. \qquad (19c)$$

The vector potential is given by

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_n f_n(t, z) \boldsymbol{F}_n(\boldsymbol{r}_t) + \sum_n g_n(t, z) \boldsymbol{G}_n(\boldsymbol{r}_t) + \sum_n \boldsymbol{a}_z a_n(t, z) \Psi_n(\boldsymbol{r}_t). \qquad (20)$$

Scalar Potential Expansion

The scalar potential $\Phi(\boldsymbol{r}, t)$ is a solution of (3) and must satisfy the boundary condition $\Phi = 0$ on C. The eigenfunction $\Psi_n(\boldsymbol{r}_t)$ are appropriate for expanding the transform $\Phi(\boldsymbol{r}_t, \beta, \omega)$ of $\Phi(\boldsymbol{r}, t)$. Thus let

$$\Phi(\boldsymbol{r}_t, \beta, \omega) = \sum_n b_n(\omega, \beta) \Psi_n(\boldsymbol{r}_t). \qquad (21)$$

By standard procedures it is then found that

$$b_n(\omega, \beta) = \frac{-q}{\epsilon_0(k_c^2 - k_n^2)} \int_{-\infty}^{\infty} \Psi_n[\boldsymbol{r}_t'(t')] e^{-j\omega t' + j\beta z'} dt'. \qquad (22)$$

Inversion with respect to β gives

$$b_n(\omega, z) = \frac{-jq}{2\epsilon_0 \beta_n} \int_{-\infty}^{\infty} \Psi_n[\boldsymbol{r}_t'(t')] e^{-j\beta_n|z-z'| - j\omega t'} dt'. \qquad (23)$$

A further inversion with respect to ω gives

$$b_n(t, z) = \frac{c^2 Z_0 q}{2} \int_{-\infty}^{\infty} \Psi_n(\boldsymbol{r}_t') J_0 \cdot [k_n \sqrt{c^2(t-t')^2 - (z-z')^2}] dt' \qquad (24)$$

where r_t' and z' are functions of t'. As a function of r and t the scalar potential is given by

$$\Phi(r, t) = \sum_n b_n(t, z)\Psi_n(r_t). \tag{25}$$

The two modal expansions (20) and (25) are the extension of the free space Liénard-Wiechert potentials, and permit the calculation of the radiation from a single arbitrarily moving charge in a uniform empty waveguide. The particle trajectory as described by $r'(t')$ must of course, be known before the expansion coefficients can be evaluated.

The electromagnetic field is given by

$$H = \frac{1}{\mu_0} \nabla \times A \tag{26}$$

$$E = -\frac{\partial A}{\partial t} - \nabla\Phi. \tag{27}$$

In a gauge where $\nabla \cdot A \neq 0$ the scalar potential may be eliminated by use of the Lorentz condition

$$\nabla \cdot A = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t} \tag{28}$$

except for that part of Φ which is independent of time. For the time dependent electric field we now have

$$E = -\frac{\partial A}{\partial t} + c^2 \int^t \nabla\nabla \cdot A dt. \tag{29}$$

It is easily verified that the vector and scalar potentials as given by (20) and (25), respectively, satisfy the Lorentz condition.

III. Electric and Magnetic Fields

In this section we present the solutions for the Fourier transforms of the electric and magnetic fields. The field can be described in terms of E and H modes. The H modes arise from the F_n functions, while the G_n and Ψ_n give rise to E modes. Since the F_n are derived from scalar functions ϕ_n that satisfy Neumann boundary conditions the dominant mode, i.e., the mode with the lowest cutoff frequency, is an H mode.

Using (26) and (27), we may obtain expressions for $H_z(r, \omega)$ and $E_z(r, \omega)$. The transverse field components $H_t(r, \omega)$ and $E_t(r, \omega)$ are then readily found from the axial field components by conventional waveguide theory. Thus we obtain for H modes the results

$$H_z(r, \omega) = \sum_n f_n(\omega, z)\frac{l_n}{\mu_0}\phi_n(r_t) \tag{30a}$$

$$H_t(r, \omega) = \frac{1}{l_n^2}\nabla_t \frac{\partial H_z}{\partial z} = \sum_n \frac{1}{l_n \mu_0}\frac{\partial f_n(\omega, z)}{\partial z}\nabla_t \phi_n(r_t) \tag{30b}$$

$$E_t(r, \omega) = \frac{jk_0 Z_0}{\mu_0}\sum_n \frac{f_n(\omega, z)}{l_n} a_z \times \nabla_t \phi_n(r_t). \tag{30c}$$

For the E modes we obtain

$$E_z(r, \omega) = \sum_n \frac{k_n Z_0}{jk_0\mu_0}\left[k_n a_n(\omega, z) - \frac{\partial g_n(\omega, z)}{\partial z}\right]\Psi_n(r_t) \tag{31a}$$

$$E_t(r, \omega) = \frac{Z_0}{jk_0\mu_0}\sum_n \frac{1}{k_n} \cdot \left[k_n \frac{\partial a_n(\omega, z)}{\partial z} + \beta_n^2 g_n(\omega, z)\right]\nabla_t \Psi_n(r_t) \tag{31b}$$

$$H_t(r, \omega) = \frac{1}{\mu_0}\sum_n \frac{1}{k_n} \cdot \left[\frac{\partial g_n(\omega, z)}{\partial z} - k_n a_n(\omega, z)\right] a_z \times \nabla_t \Psi_n(r_t). \tag{31c}$$

The expansion coefficients $f_n(\omega, z)$, $g_n(\omega, z)$, and $a_n(\omega, z)$ are given by (17).

IV. Spectral Density Distribution of Radiation

The two most important characteristics of the radiation from moving charges are the intensity and the spectral distribution (power radiated as a function of ω). The amplitude spectral distribution may be found from the expansions for $A(r, \omega)$, $\Phi(r, \omega)$ in terms of the expansion coefficients $a_n(\omega, z)$, $f_n(\omega, z)$, $g_n(\omega, z)$, and $b_n(\omega, z)$ as given by (17) and (23). However, the closed form evaluation of the expansion coefficients would, in general, be difficult and approximations must often be resorted to.

The total energy radiated in one direction in the waveguide is given by

$$W_t = \int_{-\infty}^{\infty}\int_S E_t(r, t) \times H_t^*(r, t) \cdot dS dt \tag{32a}$$

where S is a cross-sectional plane in the waveguide with $|z| \gg |z'|$ for all z'. Since the fields are real functions $H_t^* = H_t$. In terms of the Fourier transforms of the fields we obtain

$$W_t = \frac{1}{4\pi^2}\int_S \int\int_{-\infty}^{\infty}\int E_t(r, \omega) \times H_t^*(r, \lambda) \cdot dS e^{j(\omega-\lambda)t} d\omega d\lambda dt$$
$$= \int_{-\infty}^{\infty}\int_S E_t(r, \omega) \times H_t^*(r, \omega) \cdot dS \frac{d\omega}{2\pi} \cdot \tag{32b}$$

The quantity

$$W(f) = \int_S E_t(r, \omega) \times H_t^*(r, \omega) \cdot dS \tag{33}$$

is the energy spectral density, since $W(f)df$ is the energy radiated in the frequency interval df. The total energy radiated in one direction in the waveguide is given by

$$W_t = \int_{-\infty}^{\infty} W(f) df. \tag{34}$$

Since the E and H modes are orthogonal, the energy spectral densities may be computed separately for each mode.[1] Thus for H modes we obtain

$$W = \frac{k_0 Z_0}{\mu_0^2} \sum_n p_n f_n(\omega, z) f_n^*(\omega, z). \qquad (35)$$

The summation is taken over the propagating modes only.

For the E modes we find that

$$W = \frac{Z_0}{k_0 \mu_0^2} \sum_n \beta_n \left| k_n a_n(\omega, z) \pm j\beta_n g_n(\omega, z) \right|^2 \qquad (36)$$

where the upper and lower signs refer to radiation in the $\pm z$ directions and the summation is taken over the propagating modes only.

It should be noted that a considerable amount of energy may be radiated and stored within the waveguide, in the vicinity of the moving charge, in the form of evanescent waveguide modes. This stored energy is not evaluated by (32). That is, W_t gives only the total radiated energy propagating in the waveguide at large values of $|z-z'|$.

Examination of (17) for the expansion coefficients $f_n(\omega, z)$, $g_n(\omega, z)$, and $a_n(\omega, z)$ shows that they become infinite at the cutoff frequencies where p_n and β_n vanish, i.e., when $k_0 = l_n$ or k_n. This is a resonance phenomena that occurs when a mode ceases to propagate. The waveguide now behaves as a resonant cavity and the field is a standing wave field along the transverse directions in the waveguide. In a physical waveguide these infinities do not occur because of the finite conductivity which limits the Q, and hence the response, of the waveguide. By a suitable perturbation analysis, modified expressions for p_n and β_n may be derived so that attenuation due to finite conductivity is accounted for [10]. It is found that the perturbed propagation constants have the form $jp_n + j\delta_n + \delta_n$, and $j\beta_n + j\alpha_n + \alpha_n$, and do not vanish. However, at cutoff the energy spectral density will be large since the attenuation constants δ_n and α_n are small when the only losses present are caused by the finite conductivity of the waveguide. In the case of a waveguide filled with plasma, the losses due to the plasma would be much more significant. This point is discussed again later on.

V. Periodic Particle Motion

The case of periodic particle motion is of considerable interest since this includes cyclotron radiation from a plasma. The special case of cyclotron radiation from a particle executing circular motion in a transverse plane in a circular guide has been treated by Parzen and Nomicos [11]. This solution is included as a special case in the general solution presented in Sections II to IV. In this section general formulas will be derived for the amplitude spectral distribution and the average power radiated per revolution of the charge.

Consider a periodic function $h(t')$ with period $\tau = 2\pi/\omega_c$. $h(t')$ may be represented by a Fourier series

$$h(t') = \sum_{m=-\infty}^{\infty} h(\omega_m) e^{j\omega_m t'}, \qquad \omega_m = m\omega_c \qquad (37a)$$

where

$$h(\omega_m) = \frac{1}{\tau} \int_0^\tau h(t') e^{-j\omega_m t'} dt'. \qquad (37b)$$

The Fourier transform of $h(t')$ is then given by

$$h(\omega) = \sum_{m=-\infty}^{\infty} h(\omega_m) 2\pi\delta(\omega - \omega_m). \qquad (38)$$

The power or average energy per period is given by

$$\frac{1}{\tau}\int_0^\tau h(t')h^*(t')dt' = \frac{1}{\tau}\int_0^\tau \sum_m \sum_s h(\omega_m) h^*(\omega_s) e^{j(\omega_m - \omega_s)t} dt = \sum_m h(\omega_m) h^*(\omega_m) \qquad (39)$$

since the time average is zero unless $\omega_s = \omega_m$.

If the periodic motion exists for a finite time T only, say T_1 to T_2, the time function is given by

$$h(t')[U(t - T_1) - U(t - T_2)] \qquad (40)$$

where $U(t - T_i)$ is the unit step function and is equal to zero for $t < T_i$ and equal to unity for $t \geq T_i$. The Fourier transform of (40) may be evaluated by means of the convolution theorem and is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(\omega - \lambda) U(\lambda) d\lambda$$

where $U(\lambda)$ is the Fourier transform of $U(t - T_1) - U(t - T_2)$. Evaluating $U(\lambda)$ and using (38) we obtain

$$\mathfrak{F}h(t')[U(t - T_1) - U(t - T_2)] = \sum_{m=-\infty}^{\infty} T h(\omega_m) e^{-j(\omega - \omega_m)(T_1 + T_2)/2} \frac{\sin(\omega - \omega_m)T/2}{(\omega - \omega_m)T/2}. \qquad (41)$$

This result gives the well-known line broadening due to the periodic motion existing for a finite time T only. If T is greater than about 10τ, the broadened spectrum of each harmonic does not overlap the adjacent harmonics by any significant amount. In this case the average energy per period is still given essentially by (39).

The energy spectral density may thus be expressed as

$$W(f) = \sum_{m=-\infty}^{m} T h(\omega_m) h^*(\omega_m) \delta(f - f_m). \qquad (42)$$

The previously mentioned basic results from Fourier transform theory may be applied directly to the expan-

[1] An exception occurs for degenerate modes which may become coupled together by the finite losses in the waveguide. In this case new modes that are linear combinations of the old may be introduced so that the degeneracies are split and power orthogonality is still maintained, (see Gustincic [10]).

sions for the potentials. The Fourier coefficients for the mth harmonic are given by [see (17)]

$$a_n(\omega_m, z) = -\frac{j\mu_0 q}{2\beta_n \tau}\int_0^\tau \Psi_n[r_t'(t')]\frac{dz'}{dt'} \cdot e^{-j\beta_n|z-z'|}e^{-j\omega_m t'}dt' \quad (43a)$$

$$f_n(\omega_m, z) = -\frac{j\mu_0 q}{2p_n \tau}\int_0^\tau F_n(r_t')\cdot\frac{dr_t'}{dt'} \cdot e^{-jp_n|z-z'|}e^{-j\omega_m t'}dt' \quad (43b)$$

$$g_n(\omega_m, z) = -\frac{j\mu_0 q}{2\beta_n \tau}\int_0^\tau G_n(r_t')\cdot\frac{dr_t'}{dt'} \cdot e^{-j\beta_n|z-z'|}e^{-j\omega_m t'}dt' \quad (43c)$$

where β_n and p_n are evaluated at $\omega=\omega_m=m\omega_c$.

If the periodic motion persists for an interval T long compared with the period τ the power P radiated is given by the following:

For H modes

$$P = \frac{W_t}{T} = \frac{Z_0}{\mu_0^2}\sum_{m=-\infty}^{\infty}\sum_n p_n k_0 f_n(\omega_m, z) f_n^*(\omega_m, z) \quad (44)$$

while for E modes

$$P = \frac{Z_0}{\mu_0^2}\sum_{m=-\infty}^{\infty}\sum_n \frac{\beta_n}{k_0}\left| k_n a_n(\omega_m, z) \pm j\beta_n g_n(\omega_m, z)\right|^2 \quad (45)$$

where k_0, β_n, and p_n are evaluated for $\omega=m\omega_c=\omega_m$ for the mth harmonic. The upper and lower signs in (45) refer to the power radiated in the $\pm z$ directions, and the sum over n is taken over the propagating modes only. Since all modes are cut off at $\omega=0$ the term $m=0$ in (44) and (45) does not contribute to the radiated power.

Cyclotron Radiation in a Rectangular Waveguide

As an example of the application of the formulas previously mentioned, consider the radiation from a charge q undergoing circular motion in the transverse plane inside a rectangular waveguide as in Fig. 1. Let the position vector of the charge be

$$\mathbf{r}'(t') = \mathbf{a}_x x_0 + \mathbf{a}_y y_0 + \frac{v}{\omega_c}(\mathbf{a}_x \cos\omega_c t' + \mathbf{a}_y \sin\omega_c t')$$

where v/ω_c is the radius of the orbit. We will consider the excitation of the dominant H_{10} mode only.

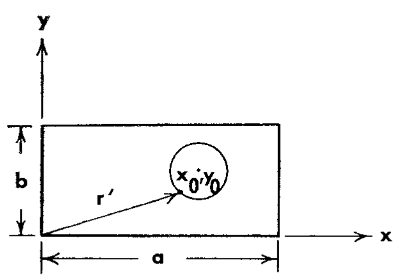

Fig. 1. Charge q n a circular orbit in a transverse plane in a rectangular waveguide.

The required scalar function ϕ_1 for the H_{10} mode is

$$\phi_1 = \sqrt{\frac{2}{ab}}\cos\frac{\pi x}{a}$$

from which $\mathbf{F}_1$ is found to be

$$\mathbf{F}_1 = \sqrt{\frac{2}{ab}}\,\mathbf{a}_y \sin\frac{\pi x}{a}\cdot$$

From (43b) we obtain

$$f_1(\omega_m, z) = -\frac{\mu_0 qj}{2p_1\tau}\sqrt{\frac{2}{ab}}e^{-jp_1|z|}\int_0^\tau \sin\frac{\pi}{a} \cdot\left(x_0 + \frac{v}{\omega_c}\cos\omega_c t'\right) v e^{-j\omega_m t'}\cos\omega_c t' dt'$$

where $p_1=[(m\omega_c/c)^2-(\pi/a)^2]^{1/2}$ and $z'=0$. Using a well-known expansion of

$$\sin\frac{\pi}{a}\left(x_0 + \frac{v}{\omega_c}\cos\omega_c t'\right),$$

we obtain

$$f_1(m\omega_c, z) = \frac{j\mu_0 qv}{2p_1}\sqrt{\frac{2}{ab}}e^{-jp_1|z|} \cdot\begin{Bmatrix} (-1)^{(m+1)/2}J_m'\sin\dfrac{\pi x_0}{a}, & m \text{ odd} \\ (-1)^{m/2}J_m'\cos\dfrac{\pi x_0}{a}, & m \text{ even}\end{Bmatrix} \quad (46)$$

where $J_m'=dJ_m(\gamma)/d\gamma$, $\gamma=\pi v/\omega_c a$, and J_m is the mth order Bessel function.

The power radiated in one direction as an H_{10} mode at the mth harmonic is found from (44) to be

$$P = \frac{Z_0 q^2 m\omega_c v^2}{4cp_1}(J_m')^2\frac{2}{ab}\begin{Bmatrix}\sin^2\dfrac{\pi x_0}{a}, & m \text{ odd} \\ \cos^2\dfrac{\pi x_0}{a}, & m \text{ even}\end{Bmatrix}. \quad (47)$$

The $-m$ harmonic will contribute a power equal to P also. If we have a uniform distribution of particles over the guide cross section and these radiate incoherently the average power radiated per particle is

$$P_a = \frac{1}{ab}\int_0^a\int_0^b P dx_0 dy_0 = \frac{Z_0 q^2 v^2 m\omega_c}{4c\sqrt{\left(\dfrac{m\omega_c}{c}\right)^2 - \left(\dfrac{\pi}{a}\right)^2}}\left[J_m'\left(\frac{\pi v}{\omega_c a}\right)\right]^2. \quad (48)$$

In a plasma an average over the velocity distribution of the particles would also be taken.

VI. Radiation from Accelerated Charges in Plasma-Filled Waveguides

The preceding analysis has been based on the assumption that a particle radiates into an empty waveguide. The mode expansions used were, therefore, the ones appropriate to an empty waveguide. In calculating the radiation from a single particle in a waveguide filled with plasma, the collective action of the plasma will modify the radiation from a single particle. From a macroscopic viewpoint the only change in the analysis that is required is the use of modal expansions appropriate to the plasma-filled waveguide. Since in a waveguide filled with plasma (or nonuniformly filled) the equations satisfied by the potentials are much more involved than those given by (2) and (3), it is often more expedient to work directly with the modal expansions for the electric and magnetic fields.

Consider a waveguide completely or partially filled with a plasma that is uniform in the axial z direction. A general solution for the propagating modes in this type of waveguide, taking into account an applied magnetic field, nonuniform density over the cross section, temperature effects, etc., has not yet been developed. Only certain simplified situations are amenable to mathematical analysis. These are based on the assumption of a temperate plasma and uniform density over the cross section of the plasma, e.g,. plasma slabs and columns. With these assumptions the plasma may be characterized as an anisotropic (gytropic) medium with a tensor dielectric constant. The density may vary over the cross section and this may be taken into account by considering the tensor dielectric constant to be a function of the transverse coordinates. The solutions for the fields for this type of plasma model is summarized in considerable detail by Bers [12].

For the purpose of the present section we shall assume that it has been possible to find the complete set of free modes describing the electric and magnetic fields in the plasma-filled guide that is of interest. The modes of the plasma-filled guide are then represented as follows:

$$E_n^+(r, \omega) = [E_{tn}^+(r_t, \omega) + E_{zn}^+(r_t, \omega)]e^{-j\beta_n^+ z} \quad (49a)$$

$$H_n^+(r, \omega) = [H_{tn}^+(r_t, \omega) + H_{zn}^+(r_t, \omega)]e^{-j\beta_n^+ z} \quad (49b)$$

$$E_n^-(r, \omega) = [E_{tn}^-(r_t, \omega) + E_{zn}^-(r_t, \omega)]e^{j\beta_n^- z} \quad (49c)$$

$$H_n^-(r, \omega) = [H_{tn}^-(r_t, \omega) + H_{zn}^-(r_t, \omega)]e^{j\beta_n^- z} \quad (49d)$$

where the $+$ and $-$ signs refer to modes propagating in the $+z$ and $-z$ directions. If the waveguide has reflection symmetry about a z=constant plane, then $\beta_n^- = \beta_n^+$ and $E_{zn}^- = -E_{zn}^+$, $H_{tn}^- = -H_{tn}^+$. Reflection symmetry will occur if the applied static magnetic field is along the waveguide axis z. If the magnetic field is applied in a direction perpendicular to z, the waveguide does not exhibit reflection symmetry and β_n^- does not equal β_n^+ in general.

To have available an orthogonality property for the modes in a waveguide characterized by a nonsymmetric dielectric tensor $[\kappa]$ it is necessary to introduce the modes that are solutions for the same waveguide but with media characterized by the transposed tensor $[\tilde{\kappa}]$, [13], [14]. The modes obtained in the guide with media described by $[\tilde{\kappa}]$ will be denoted by the same symbols as in (49) with the addition of a tilde ($\sim$), e.g., $\tilde{E}_n(r, \omega)$. Similarly, the propagation constants are denoted by $\tilde{\beta}_n^\pm$. It may be shown that $\tilde{\beta}_n^\pm = \beta_n^\mp$, i.e., the eigenvalues for the guide with a transposed tensor $[\kappa]$ are $-\beta_n^+$ and $-\beta_n^-$. However, no simple relationship exists between $E_n^\pm$ and $\tilde{E}_n^\pm$ [15]. The applicable orthogonality relation is

$$\int_S [E_{tn}^\pm(r_t, \omega) \times H_{tm}^\pm(r_t, \omega) - \tilde{E}_{tm}^\pm(r_t, \omega) \times H_{tn}^\pm(r_t, \omega)] \cdot a_z dS = 0 \quad (50)$$

where the integration is over the guide cross section and n, m take on all possible values. In addition we have

$$\int_S (E_{tn}^\pm \times H_{tn}^\mp - \tilde{E}_{tn}^\mp \times H_{tn}^\pm) \cdot a_z dS = \pm 1 \quad (51)$$

provided the modes are suitably normalized. For degenerate eigenvalues it is assumed that the corresponding subset of degenerate modes are linearly combined into a new subset such that (50) and (51) are still valid.

We will now consider the expansion of the propagating field radiated by an arbitrarily moving charged particle in a waveguide. The particle is assumed to remain within a finite region existing from z_1 to z_2 in the waveguide as in Fig. 2. The Fourier transform of the current corresponding to the moving charge is given by (5) as

$$J(r, \omega) = q \int_{-\infty}^{\infty} \frac{dr'(t')}{dt'} e^{-j\omega t'} \delta[r_t - r_t'(t')]\delta[z - z'(t')]dt'.$$

In the frequency domain the field radiated by this current may be expanded in terms of the modes given by (51). Thus let[2]

$$E^+(r, \omega) = \sum_n a_n E_n^+(r, \omega) \qquad z \geq z_2 \quad (52a)$$

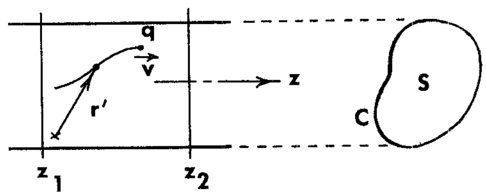

Fig. 2. A moving charged particle in a plasma filled waveguide.

[2] If any $d\beta_n^+/d\omega$ are negative, the group and phase velocities are oppositely directed. In this case the mode carries power in the $-z$ direction and is included in the expansion for $z \leq z_1$ and not for $z \geq z_2$. Similar remarks apply for the β_n^-.

$$H^+(r,\omega)=\sum_n a_n H_n^+(r,\omega)$$

$$E^-=\sum_n b_n E_n^- \qquad z\le z_1. \tag{52b}$$

$$H^-=\sum b_n H_n^-$$

The total field E, H radiated by J is a solution of

$$\nabla\times E=-j\omega\mu_0 H$$
$$\nabla\times H=j\omega\epsilon_0[\kappa]\cdot E+J.$$

The free modes in the guide characterized by $[\tilde{\kappa}]$ are solutions of

$$\nabla\times \tilde{E}_n^\pm=-j\omega\mu_0\tilde{H}_n^\pm$$
$$\nabla\times \tilde{H}_n^\pm=j\omega\epsilon_0[\tilde{\kappa}]\cdot\tilde{E}_n^\pm.$$

From these equations we obtain

$$\nabla\cdot[E\times\tilde{H}_n^\pm-\tilde{E}_n^\pm\times H]=J\cdot\tilde{E}_n^\pm.$$

If we integrate over the volume of the guide between cross-sectional planes at $z=z_1$, z_2, convert the integral of the divergence to a surface integral, note that the integral over the perfectly conducting waveguide walls vanishes, and finally, make use of (50) and (51) we obtain

$$a_n=\int_V J\cdot\tilde{E}_n^-dV \tag{53a}$$

$$b_n=\int_V J\cdot\tilde{E}_n^+dV. \tag{53b}$$

In expanded form the expansion coefficients a_n are given by

$$\begin{aligned}a_n(\omega)&=q\int_V\int_{-\infty}^{\infty}\left[\tilde{E}_{tn}^-(r_t,\omega)\cdot\frac{dr_t(t')}{dt'}+\tilde{E}_{zn}^-(r_t,\omega)\frac{dz'(t')}{dt'}\right]e^{-j\omega t'}e^{+j\beta_n^+z}\delta[r-r'(t')]dt'dV\\&=q\int_{-\infty}^{\infty}\left[\tilde{E}_{tn}^-(r_t',\omega)\cdot\frac{dr_t'}{dt'}+\tilde{E}_{zn}^-(r_t',\omega)\frac{dz'}{dt'}\right]\cdot e^{-j\omega t'+j\beta_n^+z'}dt'\end{aligned} \tag{54}$$

where r_t' and z' are functions of t' and $\tilde{\beta}_n^-=\beta_n^+$. A similar expression holds for $b_n(\omega)$. To find the fields as a function of time t the inverse Fourier transform with respect to ω must be taken. The inversion is generally difficult to carry out since the mode functions $\tilde{E}_n^\pm(r,\omega)$ are complicated functions of ω, a situation which is quite different from that encountered for radiation into an empty waveguide. However, the frequency spectrum of the radiation provides essentially all of the required information. A further complication which arises in the present case is that the modes are not orthogonal as regards power flow for two reasons, namely because of losses in the plasma and because of the anisotropic nature of the plasma. However, if only one or two propagating modes are excited the lack of power orthogonality does not introduce significant computational difficulty.

In view of the overall complexity of the problem of radiation into a plasma-filled waveguide the analysis will not be pursued any further in the present paper.

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